Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Round 1 - Algebraic Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Jacob and Tyler are initially 200 feet apart. At the same moment, Jacob begins running away from Tyler at a speed of 6 feet per second, while Tyler starts running toward Jacob at a speed of 10 feet per second. How many seconds will it take for Tyler to catch up to Jacob?

2. Alice is 12 years older than Juliet. If the product of their ages is 589, what is the sum of their ages?

3. A water tank can be filled by Pipe A in 6 hours and by Pipe B in 8 hours. If both pipes are opened together for 2 hours, and then Pipe B is closed, how long will it take for just Pipe A to fill the rest of the tank? Give your answer as a fraction.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Round 2 - Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the least common multiple of 168 and 63.

2. Find the largest 5 digit number divisible by 3 with digits that multiply to 1890.

3. Find the sum of all positive integers n such that $n^2 + n + 42$ is divisible by $n + 7$.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Worcester County Mathematics League

Junior Varsity Meet 2 - January 15, 2024

Round 3 - Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Compute

$$\frac{0.14}{\frac{1}{3}\%}$$

2. Jean is getting ice cream from a machine with two dispensers. The first dispenser serves ice cream that is 20% vanilla and 80% chocolate. The second dispenser serves ice cream that is 70% vanilla and 30% chocolate. What percentage of her ice cream should come from the first dispenser so that the final mixture is 48% vanilla?

3. In a bag of marbles, $x\%$ of the marbles are small, and the remaining marbles are large. Among the small marbles, $x\%$ are red, and the rest are blue. Among the large marbles, $2x\%$ are red and the rest are blue. The number of blue marbles is 7128, and the number of large red marbles exceeds the number of small blue marbles by 2772. How many marbles are there in total?

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Round 4 - Set Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. In a group of 130 students, each student was asked whether they play a sport and whether they play an instrument. It was found that 52 students do not play a sport, and 39 students do not play an instrument. Additionally, 16 students neither play a sport nor an instrument. How many students play both a sport and an instrument?

2. For this problem, the universal set is the set of all positive integers. Let A be the set of positive square numbers less than 1000000, and let B be the set of positive cube numbers less than 1000000. Compute

$$|(A \cap B^C) \cup (A^C \cap B)|$$

where X^C is the complement of X and $|X|$ is the size of X.

3. How many subsets of $\{1, 2, \ldots, 10\}$ are there that do not contain any pair of numbers differing by 4?

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the remainder when

$$10! = 1 \times 2 \times 3 \times \cdots \times 10$$

is divided by 11?

2. Solve for x.

$$\frac{3}{2x-6} = \frac{7}{x+10}$$

3. Fred the frog is jumping up a flight of 10 stairs. Fred can jump up either 1, 2, or 3 stairs at a time. How many possible sequences of jumps are there for Fred to get up exactly 10 stairs?

4. Find the sum of all two digit numbers $\overline{AB}$ such that the sum of its digits plus the product of its digits is exactly 8 less than the number itself.

5. What is the maximum possible area of a triangle formed from 3 vertices of a regular octagon with side length 1?

6. Compute

$$\frac{1+2+3+\cdots+18}{1-2+4-8+\cdots+256}$$

7. Suppose the prime factorization of $n!$ is of the form

$$2^a \cdot 3^b \cdot 5^c \ldots$$

If $a+b=36$, what is n?

8. I have 11 coins (quarters, dimes, nickels, pennies) that total to \$1.42. How many dimes do I have?

Worcester County Mathematics League

Junior Varsity Meet 2 - January 15, 2024

Answer Key

Round 1 - Algebraic Word Problems

1. 50
2. 50
3. $\frac{5}{2}$

Round 2 - Number Theory

1. 504
2. 76533
3. 159

Round 3 - Fractions...

1. 42
2. 44%
3. 13750

Round 4 - Set Theory

1. 55
2. 1080
3. 225

Team Round

1. 10
2. $\frac{72}{11}$
3. 274
4. 171
5. $\frac{4+3\sqrt{2}}{4}$
6. 1
7. 27
8. 3

<u>Worcester County Mathematics League</u>
Junior Varsity Meet 2 - January 15, 2024
Round 1 - Algebraic Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Jacob and Tyler are initially 200 feet apart. At the same moment, Jacob begins running away from Tyler at a speed of 6 feet per second, while Tyler starts running toward Jacob at a speed of 10 feet per second. How many seconds will it take for Tyler to catch up to Jacob?

Solution: The distance between Jacob and Tyler decreases at $10 - 6 = 4$ feet per second, so it takes $\frac{200}{4} = \boxed{50}$ seconds for Tyler to catch up to Jacob.

2. Alice is 12 years older than Juliet. If the product of their ages is 589, what is the sum of their ages?

Solution: Let Juliet's age be j and Alice's age be $j + 12$. Then,

$$j(j + 12) = 589 \Rightarrow j^2 + 12j - 589 = 0$$

Using the quadratic formula

$$j = \frac{-12 \pm \sqrt{12^2 + 4 \cdot 589}}{2} = \frac{-12 \pm \sqrt{2500}}{2} = \frac{-12 \pm 50}{2} = -31, 19$$

Then, Juliet must be 19 years old, and Alice must be 31 years old, giving a sum of $\boxed{50}$.

3. A water tank can be filled by Pipe A in 6 hours and by Pipe B in 8 hours. If both pipes are opened together for 2 hours, and then Pipe B is closed, how long will it take for just Pipe A to fill the rest of the tank? Give your answer as a fraction.

Solution: Pipe A fills the tank at $\frac{1}{6}$ of the tank per hour, and Pipe B fills the tank at $\frac{1}{8}$ of the tank per hour. Combined, they fill the tank at $\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$ of the tank per hour. After 2 hours, $\frac{7}{12}$ of the tank will be filled, leaving $\frac{5}{12}$ to be filled by Pipe A. This will take Pipe A $\frac{\frac{5}{12}}{\frac{1}{6}} = \boxed{\frac{5}{2}}$ hours.

Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Round 2 - Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the least common multiple of 168 and 63.

Solution: The prime factorization of 168 is $2^3 \cdot 3 \cdot 7$. The prime factorization of 63 is $3^2 \cdot 7$. Thus, the least common multiple is $2^3 \cdot 3^2 \cdot 7 = \boxed{504}$.

2. Find the largest 5 digit number divisible by 3 with digits that multiply to 1890.

Solution: The prime factorization of 1890 is $2 \cdot 3^3 \cdot 5 \cdot 7$. We see that both 5 and 7 must be digits. To maximize the number, we want the th leftmost digit to be 9, but we see that the last two remaining digits must multiply to 6. However, for the number to by divisible by 3, the sum of the digits must be divisible by 3, but this is impossible in this case no matter how we factor 6. Our next best option is to factor a 6 out of $2 \cdot 3^3$, leaving the last 2 digits to multiply to 9. We see that for the sum of digits to be divisible by 3, the remaining digits must be 3 and 3. This gives the 5 digits number $\boxed{76533}$.

3. Find the sum of all positive integers n such that $n^2 + n + 42$ is divisible by $n + 7$.

Solution: We see that if $n^2 + n + 42$ is divisible by $n + 7$, then $n^2 + n + 42 - (n - 6)(n + 7) = n^2 + n + 42 - (n^2 + n - 42) = 84$ must be divisible by $n + 7$. Then, $n + 7$ must be a factor of 84. The factors of 84 greater than 7 are $12, 14, 21, 28, 42, 84$, giving possible values $n = 5, 7, 14, 21, 35, 77$, which sum to $\boxed{159}$.

Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Round 3 - Operations on Numerical Fractions, Decimals, Percents, and Percentage Word Problems

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Compute

$$\frac{0.14}{\frac{1}{3}\%}$$

Solution:

$$= \frac{14\frac{1}{100}}{\frac{1}{3}\frac{1}{100}} = \frac{14}{\frac{1}{3}} = 14 \cdot 3 = \boxed{42}$$

2. Jean is getting ice cream from a machine with two dispensers. The first dispenser serves ice cream that is 20% vanilla and 80% chocolate. The second dispenser serves ice cream that is 70% vanilla and 30% chocolate. What percentage of her ice cream should come from the first dispenser so that the final mixture is 48% vanilla?

Solution: Suppose x of the ice cream is from the first dispenser and $1 - x$ is from the second. Then,

$$20x + 70(1 - x) = 48 \Rightarrow x = \frac{22}{50} = \frac{44}{100}$$

Thus, the answer is $\boxed{44\%}$.

3. In a bag of marbles, $x\%$ of the marbles are small, and the remaining marbles are large. Among the small marbles, $x\%$ are red, and the rest are blue. Among the large marbles, $2x\%$ are red and the

rest are blue. The number of blue marbles is 7128, and the number of large red marbles exceeds the number of small blue marbles by 2772. How many marbles are there in total?

Solution: Let the total number of marbles be t. Let $a = x\%$. Then, there are a^2t small red marbles, $a(1-a)t$ small blue marbles, $2a(1-a)t$ large red marbles, and $(1-a)(1-2a)t$ large blue marbles. Then, we have

$$a(1-a)t + (1-a)(1-2a)t = 7128 \Rightarrow (1-a)^2t = 7128$$
$$2a(1-a)t - a(1-a)t = 2772 \Rightarrow a(1-a)t = 2772$$

Dividing, we get

$$\frac{1-a}{a} = \frac{7128}{2772}$$
$$\Rightarrow 2772 - 2772a = 7128a$$
$$\Rightarrow a = \frac{2772}{9900} = \frac{7}{25}$$

Substituting, we get

$$\frac{7}{25}\frac{18}{25}t = 2772$$

$$\Rightarrow t = 22 \cdot 25 \cdot 25 = \boxed{13750}$$

Worcester County Mathematics League
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Round 4 - Set Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. In a group of 130 students, each student was asked whether they play a sport and whether they play an instrument. It was found that 52 students do not play a sport, and 39 students do not play an instrument. Additionally, 16 students neither play a sport nor an instrument. How many students play both a sport and an instrument?

Solution: The number of students who do not play a sport or do not play an instrument is $52 + 39 - 16 = 75$. Thus, the number of students that play both a sport and an instrument is $130 - 75 = \boxed{55}$.

2. For this problem, the universal set is the set of all positive integers. Let A be the set of positive square numbers less than 1000000, and let B be the set of positive cube numbers less than 1000000. Compute

$$|(A \cap B^C) \cup (A^C \cap B)|$$

where X^C is the complement of X and $|X|$ is the size of X.

Solution: This is essentially the number of positive integers less than 1000000 that are squares but not cubes or are cubes but not squares. We see that numbers that are squares and cubes are sixth powers. Since $1000000 = 1000^2 = 100^3 = 10^6$. there are 999 squares, 99 cubes, and 9 sixth powers less than 1000000. The total number of squares but not cubes and cubes but not squares is

$$999 - 9 + 99 - 9 = \boxed{1080}$$

3. How many subsets of $\{1, 2, \ldots, 10\}$ are there that do not contain any pair of numbers differing by 4?

Solution: We see that only one of 3 and 7 can be in the subset. Thus, there are 3 ways to choose which of 3 and 7 are in the subset. Similarly, there are 3 ways to choose which of 4 and 8 are in the subset. Out of 1,5,9, we can have $\{\}, \{1\}, \{5\}, \{9\}, \{1, 9\}$, so there are 5 ways to choose which of them are in the subset. Similarly for 2,6,10. Thus, there are $3 \cdot 3 \cdot 5 \cdot 5 = \boxed{225}$ such subsets.

Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the remainder when

$$10! = 1 \times 2 \times 3 \times \cdots \times 10$$

is divided by 11?

Solution: We can take the remainder at every intermediate multiplication to find that the answer is $\boxed{10}$. Additionally, Wilson's Theorem states for prime p that $(p-1)!$ leaves a remainder of $p-1$ when divided by p.

2. Solve for x.

$$\frac{3}{2x-6} = \frac{7}{x+10}$$

Solution:

$$\Rightarrow 3(x+10) = 7(2x-6)$$
$$\Rightarrow 3x+30 = 14x-42$$
$$\Rightarrow 11x = 72$$
$$\Rightarrow x = \boxed{\frac{72}{11}}$$

3. Fred the frog is jumping up a flight of 10 stairs. Fred can jump up either 1, 2, or 3 stairs at a time. How many possible sequences of jumps are there for Fred to get up exactly 10 stairs?

Solution: Let a_n be the number of sequences possible to climb n stairs. We have $a_0 = 1, a_1 = 1, and a_2 = 2$. We see that for $n \geq 3$, we have $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ since the number of sequences to climb n steps if the first jump is i is a_{n-i}. Thus, we have $a_3 = 2 + 1 + 1 = 4, a_4 = 4 + 2 + 1 = 7, a_5 = 13, a_6 = 24, a_7 = 44, a_8 = 81, a_9 = 149, a_1 0 = \boxed{274}$.

4. Find the sum of all two digit numbers $\overline{AB}$ such that the sum of its digits plus the product of its digits is exactly 8 less than the number itself.

Solution: We have

$$\begin{aligned} A + B + AB + 8 &= 10A + B \\ \Rightarrow 9A - AB &= 8 \\ \Rightarrow A(9 - B) &= 8 \end{aligned}$$

Thus, $A, 9 - B$ must be factors of 8. Considering all 4 factorizations, we get the two digit numbers 11, 25, 47, 88, which sum to $\boxed{171}$.

5. What is the maximum possible area of a triangle formed from 3 vertices of a regular octagon with side length 1?

Solution: Consider regular octagon with side length 1, $ABCDEFGH$. A triangle of maximum area would be ΔACF. Dropping altitudes from H and G to AF, we see that $AF = 1 + \sqrt{2}$. The altitude from C to AF has lenght $1 + \sqrt{2}/2$. Thus, the area is

$$\frac{1}{2}(1 + \sqrt{2})(1 + \sqrt{2}/2) = \boxed{\frac{4 + 3\sqrt{2}}{4}}$$

6. Compute

$$\frac{1 + 2 + 3 + \cdots + 18}{1 - 2 + 4 - 8 + \cdots + 256}$$

Solution: The numerator is $\frac{18 \cdot 19}{2} = 171$. The denominator is $\frac{1-(-512)}{1-(-2)} = 171$. Thus, the answer is $\boxed{1}$.

7. Suppose the prime factorization of $n!$ is of the form

$$2^a \cdot 3^b \cdot 5^c \dots$$

If $a + b = 36$, what is n?

Solution: Counting up from 1, we can count how many powers of 2 and 3 we see until we reach 36. Using this method, we get that the answer is $\boxed{27}$.

8. I have 11 coins (quarters, dimes, nickels, pennies) that total to \$1.42. How many dimes do I have?

Solution: If we use 0 quarters, we need at least 16 coins to cover the rest with dimes, nickels, and pennies, so this is impossible. If we use 1 quarter, we need at least 14 coins to cover the rest, so this is impossible. If we use 2 quarters, we need at least 11 coins to cover the rest, so this is impossible. If we use 3 quarters, we need at least 9 coins to cover the rest, so this is impossible. If we use 4 quarters, the only way to cover the rest in 7 coins is by using 3 dimes, 2 nickels, and 2 pennies. If we use 5 quarters, we have to use only 2 pennies, since using 7 or more would go over 11 coins. However, it is impossible to cover the remaining 15 cents with exactly 4 dimes and nickles. Thus, the in the only possible configuration, we use $\boxed{3}$ dimes.

Worcester County Mathematics League
Junior Varsity Meet 2 - January 15, 2024
Team Round Answer Sheet

ANSWERS

1. ______________________

2. ______________________

3. ______________________

4. ______________________

5. ______________________

6. ______________________

7. ______________________

8. ______________________

Worcester County Mathematics League

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Answer Key

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1. 50
2. 50
3. $\frac{5}{2}$

Round 2 - Number Theory

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Round 3 - Fractions...

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2. 44%
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Round 4 - Set Theory

1. 55
2. 1080
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Team Round

1. 10
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5. $\frac{4+3\sqrt{2}}{4}$
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